# Chapter 16: Hints and Selected Solutions

## Section 16.1 (page 449)

**16.3** You are asked to give two distinct derivations of the ambig-wff $A_1 \to A_2 \leftrightarrow \neg A_2$. Here is one. You should be able to think of another. By the basis clause, $A_2$is an ambig-wff. Hence, by the induction clause, $\neg A_2$ is an ambig-wff. Then, using the induction clause again, $A_2 \leftrightarrow \neg A_2$ is an ambig-wff. By the basis clause, $A_1$ is an ambig-wff. Using the induction clause again gives us that $A_1 \to A_2 \leftrightarrow \neg A_2$ is an ambig-wff.

**16.7** Hint: Consider the word *noon*. Is it a palindrome? Is it a pal?

## Section 16.2 (page 453)

**16.12** The set $S$ of wffs is the smallest set satisfying the following clauses:

1. Each propositional letter is in $S$.
2. If $p$ is in $S$, then so is $\neg p$.
3. If $p$ and $q$ are in $S$, then so are $(p \wedge q)$, $(p \vee q)$, $(p \to q)$, and $(p \leftrightarrow q)$.

**16.13** Hint: Your proof will have a basis case, corresponding to clause (1) and an inductive case, corresponding to clauses (2) and (3). There will be one case for clause (2) and four for clause (3).

## Section 16.3 (page 455)

**16.14** We want to prove by induction that for each natural number $n$, $n \leq 2n$.
*Basis:* We need to prove that $0 \leq 2 \times 0$. But this is clear since $2 \times 0 = 0$ and $0 \leq 0$.
*Induction step:* We assume that $n \leq 2n$ and prove that $n+1 \leq 2(n+1)$. To prove the latter, notice that $2(n+1) = 2n + 2$. By the induction hypothesis, $2n \geq n$ so $2(n+1) \geq n+2 > n+1$. Hence $n+1 \leq 2(n+1)$ as desired.

**16.17** We want to use induction to prove that for all natural numbers $n \geq 2$,

$$(1-\frac{1}{2})(1-\frac{1}{3})\ldots(1-\frac{1}{n}) = \frac{1}{n}$$

This is a little different than a normal proof by induction, since it starts not at 0 or 1, but at $n = 2$. The reason it has to start here is that the claim is false for 0 and 1. (It is not even clear exactly what it would say for $n = 0$.) *Basic case:* This is the case $n = 2$. The equation amounts to $(1 - 1/2 = 1/2$, which is obvious.

*Induction step:* We assume that

$$(1-\frac{1}{2})(1-\frac{1}{3})\ldots(1-\frac{1}{n}) = \frac{1}{n}$$

and prove that

$$(1-\frac{1}{2})(1-\frac{1}{3})\ldots(1-\frac{1}{n+1}) = \frac{1}{n+1}$$

To prove the latter equation, note that

$$(1-\frac{1}{2})\ldots(1-\frac{1}{n+1}) = ((1-\frac{1}{2})\ldots(1-\frac{1}{n})(1-\frac{1}{n+1})$$

But by the induction hypothesis, this is $(1/n)(1-1/(n+1))$. To compute the latter, first note that

$$1-\frac{1}{n+1} = \frac{n}{n+1}$$

. This gives us our desired result by canceling $n$.

## Section 16.4 (page 458)

**16.19** A proof of 16.19 is shown below. We have left you the fun of filling in the support steps. Of course you might prefer to find a different proof altogether. (We have reformatted the fourth premise a bit, to make our proof easier to read. We hope GG does not mark us off for this!)

```
1. 0 + 1 = 1
2. ∀x (x + 0 = x)
3. ∀x ∀y [x + (y + 1) = (x + y) + 1]
4. [0 + 1 = 1 + 0 ∧
   ∀x (x + 1 = 1 + x → (x + 1) + 1 = 1 + (x + 1))] →
   ∀x (x + 1 = 1 + x)
5. 1 + 0 = 1
6. 0 + 1 = 1 + 0
   7. [b] b + 1 = 1 + b
   8. (b + 1) + 1 = (b + 1) + 1
   9. 1 + (b + 1) = (1 + b) + 1
   10. 1 + (b + 1) = (b + 1) + 1
   11. (b + 1) + 1 = 1 + (b + 1)
12. ∀x (x + 1 = 1 + x → (x + 1) + 1 = 1 + (x + 1))
13. ∀x (x + 1 = 1 + x)
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**16.20** Hint: One of the axioms tells us that $\forall x\,(x + 0 = x)$, so if we only knew that addition were commutative, this would be easy. But this is one of the steps used in 16.25 in proving that addition is commutative. What you will need to do instead is to prove $x\ (0 + x = x)$ by induction on $x$. The basis case will be showing that $0 + 0 = 0$, which follows from one of the axioms by $\forall$ **Elim**. The induction step will be to assume that $0 + x = x$ and show that $0 + (x + 1) = x + 1$. How can you justify this from the available axioms?