

Chapter 15: Hints and Selected Solutions

Section 15.1 (page 411)

- 15.1** Hint: The exercise is to test your understanding of the axiom of extensionality. According to that axiom, sets are identical if and only if they have the same members, regardless of how the members are listed. In particular, the order in which they are listed is irrelevant, as is the number of times a member gets listed. For example. $\{2, 3, 2\} = \{2, 3\} = \{3, 2\}$.
- 15.2**
1. $\{7, 11, 13\}$
 4. Hint: This will be a set with only one word in it. The word describes a kind of person with lots of buzz about them.
- 15.3**
1. 19, 29, 31, for example. You should not list the same three.
 4. *pseudo*, *un*, *pre*, for example.
- 15.5** Here is a proof of the argument:

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□  $\forall x (x \in a \leftrightarrow P(x))$ 
□  $\forall x (x \in b \leftrightarrow P(x))$ 
├─ □ c
│  └─  $c \in a$ 
│     └─  $c \in a \leftrightarrow P(c)$ 
│        └─  $P(c)$ 
│           └─  $c \in b \leftrightarrow P(c)$ 
│              └─  $c \in b$ 
│
│     └─  $c \in b$ 
│        └─  $c \in a \leftrightarrow P(c)$ 
│           └─  $c \in b \leftrightarrow P(c)$ 
│              └─  $P(c)$ 
│                 └─  $c \in a$ 
│
└─  $c \in a \leftrightarrow c \in b$ 
□  $\forall x (x \in a \leftrightarrow x \in b)$ 

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Section 15.2 (page 414)

- 15.7
1. True, since it only citizens are eligible for election to the senate.
 4. True, since John's brothers are among his relatives.
 7. False. The quotes mean that the members of the sets are numerical terms, numbers. The numeral "2" is in the first set, not in the second, for example.

15.13	<ol style="list-style-type: none"> 1. $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$ 2. $\forall x \forall y (x \subseteq y \leftrightarrow \forall z (z \in x \rightarrow z \in y))$ 3. $\forall x x \subseteq x$ 																																														
	<table border="0" style="border-left: 1px solid black; padding-left: 10px;"> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">4. a f</td> <td style="padding-left: 5px;">▼</td> <td></td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">5. $a \subseteq a$</td> <td style="padding-left: 5px;">▼</td> <td>VElim: 3</td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">6. $a = f$</td> <td style="padding-left: 5px;">▼</td> <td></td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">7.</td> <td style="padding-left: 5px;">▼</td> <td>Rule?:</td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">8. $a \subseteq f \wedge f \subseteq a$</td> <td style="padding-left: 5px;">▼</td> <td>Taut Con:</td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">9. $a \subseteq f \wedge f \subseteq a$</td> <td style="padding-left: 5px;">▼</td> <td></td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">10.</td> <td style="padding-left: 5px;">▼</td> <td>Rule?:</td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">11. $\forall z (z \in a \leftrightarrow z \in f) \rightarrow a = f$</td> <td style="padding-left: 5px;">▼</td> <td>VElim: 1</td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">12. c</td> <td style="padding-left: 5px;">▼</td> <td></td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">13.</td> <td style="padding-left: 5px;">▼</td> <td>Rule?:</td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">14. $c \in a \leftrightarrow c \in f$</td> <td style="padding-left: 5px;">▼</td> <td>\leftrightarrow Intro:</td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">15. $\forall z (z \in a \leftrightarrow z \in f)$</td> <td style="padding-left: 5px;">▼</td> <td>\forall Intro: 12-14</td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">16. $a = f$</td> <td style="padding-left: 5px;">▼</td> <td>\rightarrow Elim: 15,11</td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">17. $a = f \leftrightarrow (a \subseteq f \wedge f \subseteq a)$</td> <td style="padding-left: 5px;">▼</td> <td>\leftrightarrow Intro: 9-16,6-8</td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">18. $\forall x \forall y (x = y \leftrightarrow (x \subseteq y \wedge y \subseteq x))$</td> <td style="padding-left: 5px;">▼</td> <td>\forall Intro: 4-17</td> </tr> </table>	4. a f	▼		5. $a \subseteq a$	▼	VElim: 3	6. $a = f$	▼		7.	▼	Rule?:	8. $a \subseteq f \wedge f \subseteq a$	▼	Taut Con:	9. $a \subseteq f \wedge f \subseteq a$	▼		10.	▼	Rule?:	11. $\forall z (z \in a \leftrightarrow z \in f) \rightarrow a = f$	▼	VElim: 1	12. c	▼		13.	▼	Rule?:	14. $c \in a \leftrightarrow c \in f$	▼	\leftrightarrow Intro:	15. $\forall z (z \in a \leftrightarrow z \in f)$	▼	\forall Intro: 12-14	16. $a = f$	▼	\rightarrow Elim: 15,11	17. $a = f \leftrightarrow (a \subseteq f \wedge f \subseteq a)$	▼	\leftrightarrow Intro: 9-16,6-8	18. $\forall x \forall y (x = y \leftrightarrow (x \subseteq y \wedge y \subseteq x))$	▼	\forall Intro: 4-17	
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Section 15.3 (page 418)

15.14 We have filled in the first 11 steps of Proof Intersection 1 below:

1.	$\forall x \forall y \forall z (z \in \text{int}(x, y) \leftrightarrow (z \in x \wedge z \in y))$	
2.	$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$	
3.	c	
4.	$c \in \text{int}(a, b)$	
5.	$c \in \text{int}(a, b) \leftrightarrow (c \in a \wedge c \in b)$	✓ ▾ ∇Elim: 1
6.	$c \in a \wedge c \in b$	✓ ▾ ↔Elim: 5,4
7.	$c \in a$	✓ ▾ ∧Elim: 6
8.	$c \in b$	✓ ▾ ∧Elim: 6
9.	$c \in b \wedge c \in a$	✓ ▾ ∧Intro: 8,7
10.	$c \in \text{int}(b, a) \leftrightarrow (c \in b \wedge c \in a)$	✓ ▾ ∇Elim: 1
11.	$c \in \text{int}(b, a)$	✓ ▾ ↔Elim: 10,9

- 15.15 1. {2, 4}
 4. {} (the empty set)
 7. {2, 3, 4, 5}

15.17 We first show a proof of 15.17 under construction. Notice that at step 5 we are just about to use a default instance of **∇Elim**. The completed proof is shown below.

1.	$\forall x \forall y \forall z (z \in \text{union}(x, y) \leftrightarrow (z \in x \vee z \in y))$	
2.	$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$	
3.	c	
4.	$c \in \text{union}(a, b) \leftrightarrow (c \in a \vee c \in b)$	✓ ▾ ∇Elim: 1
5.	$:x>b :y>a :z>c$	▾ ∇Elim: 1
6.	$c \in \text{union}(a,b) \leftrightarrow c \in \text{union}(b,a)$	▾ Rule?:
7.	$\forall z (z \in \text{union}(a, b) \leftrightarrow z \in \text{union}(b, a))$	✓ ▾ ∇Intro: 3-6
8.	$\forall z (z \in \text{union}(a, b) \leftrightarrow z \in \text{union}(b, a)) \rightarrow \text{union}(a, b) = \text{union}(b, a)$	✓ ▾ ∇Elim: 2
9.	$\text{union}(a, b) = \text{union}(b, a)$	✓ ▾ →Elim: 8,7

1. $\forall x \forall y \forall z (z \in \text{union}(x, y) \leftrightarrow (z \in x \wedge z \in y))$	
2. $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$	
3. $\square \nabla$	
4. $c \in \text{union}(a, b) \leftrightarrow (c \in a \wedge c \in b)$	✓ ▽ $\forall\text{Elim: 1}$
5. $c \in \text{union}(b, a) \leftrightarrow (c \in b \wedge c \in a)$	✓ ▽ $\forall\text{Elim: 1}$
6. $c \in \text{union}(a, b) \leftrightarrow c \in \text{union}(b, a)$	✓ ▽ Taut Con: 5,4
7. $\forall z (z \in \text{union}(a, b) \leftrightarrow z \in \text{union}(b, a))$	✓ ▽ $\forall\text{Intro: 3-6}$
8. $\forall z (z \in \text{union}(a, b) \leftrightarrow z \in \text{union}(b, a)) \rightarrow \text{union}(a, b) = \text{union}(b, a)$	✓ ▽ $\forall\text{Elim: 2}$
9. $\text{union}(a, b) = \text{union}(b, a)$	✓ ▽ $\rightarrow\text{Elim: 8,7}$

15.21 We want to give an informal proof of $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$. By Proposition 3, it suffices to prove $a \cap (b \cup c) \subseteq (a \cap b) \cup (a \cap c)$ and $(a \cap b) \cup (a \cap c) \subseteq a \cap (b \cup c)$. We prove the first part, leaving the second to you.

Let $x \in a \cap (b \cup c)$. Then $x \in a$ and $x \in (b \cup c)$. Hence $x \in b$ or $x \in c$. There are two cases to consider. First assume $x \in b$. Then $x \in (a \cap b)$ and hence $x \in (a \cap b) \cup (a \cap c)$. The case where $x \in c$ is similar.

Section 15.4 (page 421)

15.26 1. If $x = y$ then the set under consideration can also be written $\{\{x\}\}$ and so has one element. On the other hand, if $x \neq y$ then $\{x\} \neq \{x, y\}$, since one has one element, the other two, so the set under consideration has two elements.

2. We know that the set exists by three applications of the Unordered Pair Theorem.

3. We want to prove that if $x = u \wedge y = v$ then $\langle x, y \rangle = \langle u, v \rangle$.

Proof: Assume that $x = u$ and $y = v$. By one application of $=$ **Elim**, we see that $\langle x, y \rangle = \langle u, y \rangle$. Another application gives us $\langle x, y \rangle = \langle u, v \rangle$, as desired.

4. We want to prove

$$\langle x, y \rangle = \langle u, v \rangle \rightarrow (x = u \wedge y = v)$$

Proof: Assume $\langle x, y \rangle = \langle u, v \rangle$. Writing out the definition of these sets, we see that $\{\{x\}, \{x, y\}\} = \{\{u\}, \{u, v\}\}$. There are two cases to consider, one where $x = y$ and one where $x \neq y$. If $x = y$ then by part (1), the set on the left has only one element, namely, $\{x\}$. But then the set on the right has the same element. Hence both $\{u\}$ and

$\{u, v\}$ but be the set $\{x\}$. But then, by extensionality, $x = u$ and $x = u = v$ so $x = y = u = v$. The second case, where $x \neq y$ is a bit more complicated. We leave it to you to think through.

Section 15.5 (page 425)

15.29 This Exercise has three goals. We give a proof of the second of the three below.

- | | |
|--|---|
| 1. | |
| 2. a b ∇ SameShape(a,b) | |
| 3. ∇ \neg SameShape(b, a) | |
| 4. \perp | ✓ ∇ Ana Con: 3,2 |
| 5. SameShape(b, a) | ✓ ∇ \neg Intro: 3-4 |
| 6. $\forall x \forall y$ (SameShape(x, y) \rightarrow SameShape(y, x)) | ✓ ∇ \forall Intro: 2-5 |

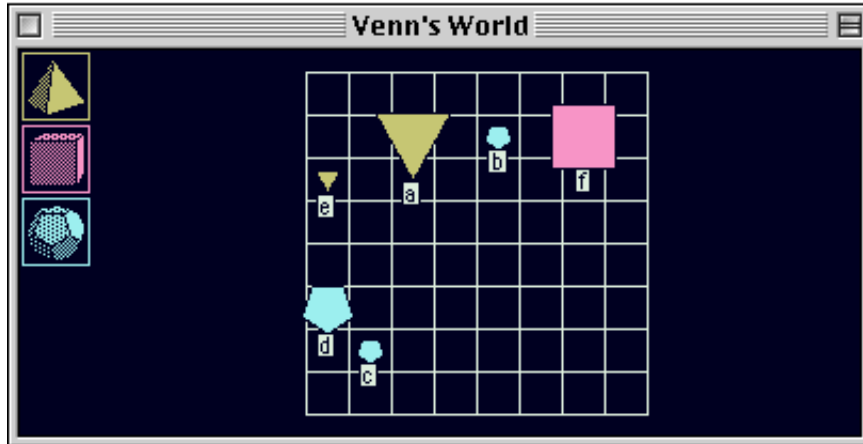
15.36

	Smaller	SameCol	Adjoins	LeftOf
Transitive	Yes			
Reflexive		Yes		
Irreflexive			Yes	
Symmetric	No	Yes		
Asymmetric				Yes
Antisymmetric	Yes			

15.38 Here is an inference scheme that goes with Symmetry:

$$\left. \begin{array}{l} R(a, b) \\ R(b, a) \end{array} \right\}$$

15.41 The 2-D view of Venn's World is shown below. The equivalence classes for *same row* are, working from front to back, $\{c\}$, $\{d\}$, $\{e\}$, and $\{a, b, f\}$.



Section 15.6 (page 428)

15.46 The 2-D view of Venn's World is shown in Figure ???. Since every block has a unique frontmost block, the function will be total. Thus, as a set of ordered pairs, it will have six elements, since there are six blocks in this world. Three of these ordered pairs are $\langle e, d \rangle$, $\langle d, d \rangle$, $\langle c, c \rangle$. What are the other three. What is the range?

15.51 The function is total since every number has a double. The range of the function is all the even natural numbers. It is one-one since if $2n = 2m$ then $n = m$.

Section 15.7 (page 431)

15.54 $\wp\{2, 3, 4\}$ has the following eight elements: $\{\}, \{2\}, \{3\}, \{4\}, \{2, 3\}, \{2, 4\}, \{2, 3\}, \{2, 3, 4\}$

15.57 Hint: The set \emptyset has exactly one subset. What is it? That will be the only member of $\wp\emptyset$.

- 15.60**
1. It's true that for any set b , $\emptyset \subseteq \wp b$. Why?
 2. You should be able to find a set b for which $b \not\subseteq \wp b$.
 3. It is not true that for any sets a and b , $\wp(a \cup b) = \wp a \cup \wp b$. Can you find a counterexample?
 4. You should be able to prove that for any sets a and b , $\wp(a \cap b) = \wp a \cap \wp b$.

15.61 Recall that Russell's set for any set b consists of those elements of b that are not elements of themselves.

1. In this case, $b = \{\emptyset\}$. It has only one element, the emptyset, which certainly is not an element of itself, so the Russell set for b is identical with b .
3. In this case, $b = \{1, a\}$ where $a = \{a\}$. One element of b is an element of itself, namely a . The other element of b , namely 1, is not an element of itself. Therefore the Russell set for this set is $\{1\}$.

Section 15.9 (page 439)

15.68 Our goal is to show that the Axiom of Separation and Extensionality are consistent. That is, we need to find a universe of discourse in which both are clearly true. As a hint, we were told to consider the domain whose only element is the empty set. This is a pretty impoverished domain of discourse, but it turns out to be one in which both of these axioms are true. Since there is only one element, the Axiom of Extensionality is true in a rather trivial way. Indeed, we have $\forall a \forall b (a = b)$ true. As for the axiom of separation, we need to show that given any property P , the set $\{x \in \emptyset \mid P(x)\}$ is in the domain of discourse. But this is again just the emptyset.

We know that there is no domain of discourse making the axioms of comprehension true, so it may seem surprising that such a trivial domain could make the closely related separation axiom, as well as extensionality, true. But really, what this shows, is just how pitifully weak this theory is without the other axioms that we have added to ZFC.

15.69 Hint: We want to modify the previous solution. The first thought is to add to the domain some set $\{p\}$, where p is not a set. This won't quite work because $\{p\} \cup \{p\} = \{p\}$ and $\{p\} \cup \emptyset = \{p\}$. However, what happens if we add both $\{p\}$, and $\{q\}$, where p and q are distinct objects that are not sets?

- 15.72**
1. If a has an absolute complement, say b , then $a \cup b$ is the universal set. But we know by a Exercise 15.67 that there is so such set.
 2. Hint: This follows easily from the axiom of separation. Just rewrite the definition in the required form.