Chapter 12: Hints and Selected Solutions

Section 12.3 (page 327)

12.1 The argument is valid and the proof is a good one. In the following, we repeat the proof, but being explicit about the proof methods being used.

$$\begin{array}{|l}
\forall \mathsf{x}\,[(\mathsf{Brillig}(\mathsf{x}) \lor \mathsf{Tove}(\mathsf{x})) \to (\mathsf{Mimsy}(\mathsf{x}) \land \mathsf{Gyre}(\mathsf{x}))] \\
\forall \mathsf{y}\,[(\mathsf{Slithy}(\mathsf{y}) \lor \mathsf{Mimsy}(\mathsf{y})) \to \mathsf{Tove}(\mathsf{y})] \\
\exists \mathsf{x}\,\mathsf{Slithy}(\mathsf{x}) \\
\hline
\exists \mathsf{x}\,[\mathsf{Slithy}(\mathsf{x}) \land \mathsf{Mimsy}(\mathsf{x})]
\end{array}$$

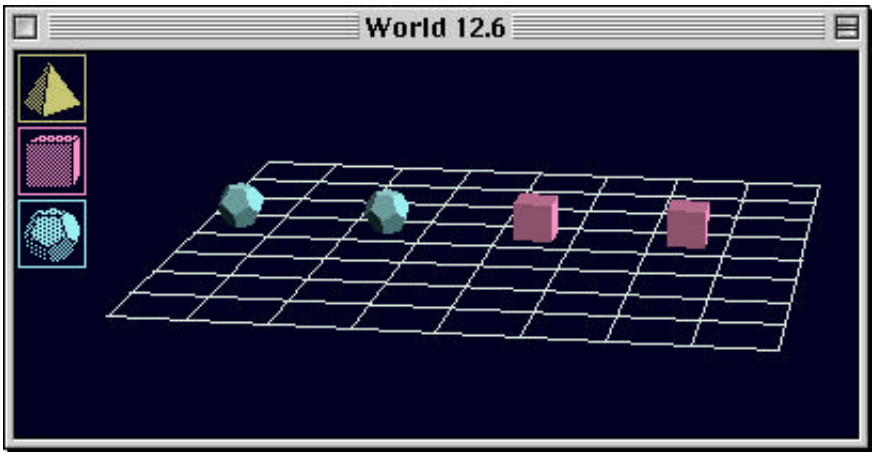
Proof: By the third premise, we know that something in the domain of discourse is slithy. In order to use $\exists$ **Elim**, let b be one of these slithy things. By the second premise, everything that is either slithy or mimsy is tove. Hence, by $\forall$ **Elim**, $\lor$ **Intro** and $\to$ **Elim**, we know that b is a tove. By the first premise, using the same methods, we see that b is mimsy and gyre. Hence b is mimsy, using $\land$ **Elim**. Thus, b is both slithy and mimsy, using $\land$ **Intro**. Hence, something is both slithy and mimsy, by $\exists$ **Intro**.

12.4 The argument is valid. We give an informal proof.

$$\begin{array}{|l}
\forall \mathsf{y}\,[\mathsf{Cube}(\mathsf{y}) \lor \mathsf{Dodec}(\mathsf{y})] \\
\forall \mathsf{x}\,[\mathsf{Cube}(\mathsf{x}) \to \mathsf{Large}(\mathsf{x})] \\
\exists \mathsf{x}\,\neg\mathsf{Large}(\mathsf{x}) \\
\hline
\exists \mathsf{x}\,\mathsf{Dodec}(\mathsf{x})
\end{array}$$

Proof: By the third premise, there is something that is not large. Let c be some such block. By the second premise, everything that is a cube is large, so c can't be a cube. But by the first premise, everything is either a cube or a dodecahedron, so c must be a dodec. Hence there is a dodecahedron.

12.6 Here's a counterexample. You should, as usual, submit a different counterexample.



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12.11 Here is the argument and the purported proof:

There is a number greater than every other number.

> **Purported proof:** Let n be an arbitrary number. Then n is less than some other number, $n+1$ for example. Let m be any such number. Thus $n \leq m$. But n is an arbitrary number, so every number is less or equal m. Hence there is a number that is greater than every other number.

The conclusion is obviously false, so there must be something wrong with the argument. The mistake comes in the step: "But n is an arbitrary number, so every number is less or equal m." This is using general conditional proof on n, but the statement being generalized contains a constant, m, introduced by $\exists$ **Elim** after n was introduced, which is not a valid inference method.

12.16 The following argument is valid:

$$\begin{array}{|l}
\forall x\, \forall y\, [\mathsf{LeftOf}(x,y) \rightarrow \mathsf{Larger}(x,y)] \\
\forall x\, [\mathsf{Cube}(x) \rightarrow \mathsf{Small}(x)] \\
\forall x\, [\mathsf{Tet}(x) \rightarrow \mathsf{Large}(x)] \\
\forall x\, \forall y\, [(\mathsf{Small}(x) \wedge \mathsf{Small}(y)) \rightarrow \neg\mathsf{Larger}(x,y)] \\
\hline
\neg\exists x\, \exists y\, [\mathsf{Cube}(x) \wedge \mathsf{Cube}(y) \wedge \mathsf{RightOf}(x,y)]
\end{array}$$

Here is an informal proof.

Proof: We want to prove

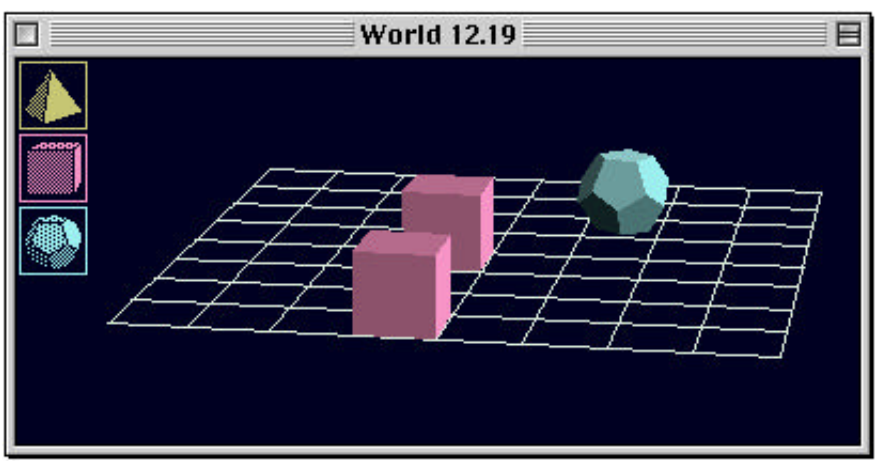
$$\neg\exists x\, \exists y\, [\mathsf{Cube}(x) \wedge \mathsf{Cube}(y) \wedge \mathsf{RightOf}(x,y)]$$

Suppose, toward a proof by contradiction, that

$$\exists x\, \exists y\, [\mathsf{Cube}(x) \wedge \mathsf{Cube}(y) \wedge \mathsf{RightOf}(x,y)]$$

Let b and c be cubes such that b is right of c. But then c is left of b. By the first premise, c is larger than b. But this contradicts the third premise, since both cubes are small.

12.19 One counterexample to the argument:



12.28 Hint: This is similar to the Barber Paradox.

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12.30 The following argument is valid.

$$\begin{array}{|l} \exists \mathsf{x}\,(\neg \mathsf{Cube}(\mathsf{x}) \wedge \neg \mathsf{Dodec}(\mathsf{x})) \\ \exists \mathsf{x}\, \forall \mathsf{y}\, \mathsf{SameShape}(\mathsf{x}, \mathsf{y}) \\ \hline \forall \mathsf{x}\, \mathsf{Tet}(\mathsf{x}) \end{array}$$

Here is a proof, one that makes explicit the use of the shape axioms.

> **Proof:** By the first premise, there is an object that is neither a cube nor a dodecahedron. Let's call some such object d. By Axiom 4, d must be a tetrahedron. By the second premise, there is a block c such that everything is the same shape as c. Hence d is the same shape as c. By Axiom 10, c is a tet. But then everything is the same shape as the tetrahedron c, so by another application of Axiom 10, everything is a tetrahedron.

12.34 We want to prove that the following argument can be justified on the basis of the ten shape axioms as given. Of course it is so obvious a principle that it might have been incorporated more directly as an axiom. But it wasn't.

$$\begin{array}{|l} \mathsf{SameShape}(\mathsf{b}, \mathsf{c}) \\ \hline \mathsf{SameShape}(\mathsf{c}, \mathsf{b}) \end{array}$$

> **Proof:** Assume $\mathsf{SameShape}(\mathsf{b}, \mathsf{c})$. By axiom 4, b must be either a tetrahedron, dodecahedron. or cube. We need to consider three cases, corresponding to these three possibilities. The proofs are similar, though, so we only consider the case where b is a dodecahedron. By axiom 9, c is also a dodecahedron. Then, by axiom 6, c is the same shape as b, that is, $\mathsf{SameShape}(\mathsf{c}, \mathsf{b})$, as desired.